

Sequences of Integers with Three Missing Separations ^{*}

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Abstract

Fix a set D of positive integers. We study the maximum density $\mu(D)$ of sequences of integers in which the separation between any two terms does not fall in D . The D -sets considered in this article are of the form $\{1, j, k\}$. The closely related function $\kappa(D)$, the parameter involved in the “lonely runner conjecture,” is also investigated. Exact values of $\kappa(D)$ and $\mu(D)$ are found for some families of $D = \{1, j, k\}$. We prove that the boundary conditions in two earlier results of Haralambis [20] are sharp. Consequently, our results declaim two conjectures posted recently in [6], and extend some results by Gupta [19].

1 Introduction

Let D be a set of positive integers. A sequence S of non-negative integers is called a D -sequence if $|x - y| \notin D$ for any $x, y \in S$. Denote $|S \cap \{0, 1, 2, \dots, n\}|$ as $S[n]$. The upper density $\bar{\delta}(S)$ and the lower density $\underline{\delta}(S)$ of S are defined, respectively, by $\bar{\delta}(S) = \overline{\lim}_{n \rightarrow \infty} S[n]/(n+1)$ and $\underline{\delta}(S) = \underline{\lim}_{n \rightarrow \infty} S[n]/(n+1)$. We say S has density $\delta(S)$ if $\bar{\delta}(S) = \underline{\delta}(S)$, and we denote this common value as $\delta(S)$. The parameter of interest is the *density of D* , $\mu(D)$, defined as

$$\mu(D) := \sup \{ \delta(S) : S \text{ is a } D\text{-sequence} \}.$$

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The parameter $\mu(D)$ is closely related to the parameter of D involved in the so called “lonely runner conjecture”. For a real number x , let $||x||$ denote the minimum distance from x to an integer, that is, $||x|| = \min\{\lceil x \rceil - x, x - \lfloor x \rfloor\}$. For any real t , denote $||tD||$ the smallest value $||td||$ among all $d \in D$. The *kappa value* of D , denoted by $\kappa(D)$, is the supremum of $||tD||$ among all real t . That is,

$$\kappa(D) := \sup\{||tD|| : t \in \mathbb{R}\}.$$

Wills [38] conjectured that $\kappa(D) \geq 1/(|D| + 1)$ is true for all finite sets D . This conjecture is also known as the *lonely runner conjecture* by Bienia et al. [2]. Suppose m runners run laps on a circular track of unit circumference. Each runner maintains a constant speed, and the speeds of all the runners are distinct. A runner is called *lonely* if the distance on the circular track between him and every other runner is at least $1/m$. Equivalently, Wills’ conjecture asserts that for each runner, there exists some time t when he becomes lonely. The conjecture has been proved true for $|D| \leq 6$ (cf. [1, 3, 13, 14]), and remains open for $|D| \geq 7$.

The problem of determining $\mu(D)$ was initially posed by Motzkin in an unpublished problem collection (cf. [4]). In 1975, Cantor and Gordon [4] demonstrated that

$$\mu(D) \geq \kappa(D), \tag{1}$$

the fundamental connection between these two concepts, by showing that a “good time”, that is a $t \in \mathbb{R}$ such that $||tD||$ is maximized, can be used to create a D -sequence. It was also proved that, for any finite set of positive integers D , $\mu(D)$ exists and this maximal density is rational and achieved by a periodic sequence with period length at most $2^{\max(D)}$. Using a different approach, Carraher et al. [6] showed a similar result but with period length at most $(\max(D) \cdot 2^{\max(D)})$. And finally, for two-element sets $D = \{a, b\}$, Cantor and Gordon [4] proved that $\kappa(D) = \mu(D) = \lfloor \frac{a+b}{2} \rfloor / (a+b)$.

For sets D with more than two elements, $\mu(D)$ and $\kappa(D)$ have been investigated intensively by researchers in different fields such as number theory and graph theory (cf. [5–12, 18, 24, 26–28, 30–32, 39, 40, 42–44]). Readers are referred to the survey [28]. For general three element sets $D = \{i, j, k\}$, Gupta [19] established lower bounds for $\mu(D)$, proved that some of them are the exact value, and conjectured others are also the exact value. If $D = \{a, b, a+b\}$ it was proved by Liu and Zhu [31] that $\kappa(D) = \mu(D)$, and the exact values were determined.

Theorem 1. [31] Suppose $D = \{a, b, a + b\}$ with $\gcd(a, b) = 1$. Then

$$\mu(D) = \kappa(D) = \max \left\{ \frac{\lfloor \frac{2b+a}{3} \rfloor}{2b+a}, \frac{\lfloor \frac{2a+b}{3} \rfloor}{2a+b} \right\}.$$

This confirmed a conjecture of Rabinowiz and Proulz [33], who had shown one direction of the equality in Theorem 1 in their study of the asymptotic behavior of the channel assignment problem. The same inequality had also been discovered by Gupta [19] independently.

For the family of sets $D = \{1, j, k\}$, the values of $\mu(D)$ were first studied by Haralambis [20]. By considering different parities of j , the author established the following two results:

Theorem 2. [20] If $D = \{1, j, k\}$, where $1 < j < k$ and j is even, and $k = n(j+1) + \bar{k}$ ($0 \leq \bar{k} \leq j$), then $\mu(D) = \frac{j}{2j+1}$ if $\bar{k} = 1$ or j . Otherwise,

$$\mu(D) = \frac{nj/2 + \lceil \bar{k}/2 \rceil}{k+1},$$

provided that $n \geq (j - \bar{k} - 2)/2$ where \bar{k} is even, and $n \geq (2j - \bar{k} - 3)/2$ where \bar{k} is odd.

Theorem 3. [20] If $D = \{1, j, k\}$, where $1 < j < k$ and j is odd, then $\mu(D) = 1/2$ if k is odd. Otherwise, $\mu(D) = \frac{k}{2(k+j)}$, provided that $k \geq j(j-1)/2$.

Using a local discharging method, Carraher et al. [6] determined the exact values of $\mu(\{1, j, k\})$ for some j and k . Among their results, the following conjectures were posed:

Conjecture 4. (Conjecture 26. [6]) Let $D = \{1, j, k\}$ where j is odd, $j \geq 3$, and k is even. Then $\mu(D) = \frac{k}{2(k+j)}$, provided that $k \geq 3j$.

Conjecture 5. (Conjecture 29. [6]) Let $D = \{1, j, k\}$, where both j and k are even, and $k > j$. Then $\mu(D) = \frac{j}{k+j}$.

Conjecture 4 was proved true when $j = 3, 5, 7$ by Carraher et al. [6]. Moreover, Theorem 3 showed it holds for $k \geq j(j-1)/2$.

In this article we show Conjecture 4 is not always true by determining the exact value of $\mu(D)$ for some families of the sets D in Conjecture 4 with $k \leq j(j-1)/2$

(Theorems 8, 10 and 11). Our results reveal that the condition $k \geq j(j-1)/2$ required in Theorem 3 is sharp.

One direction of Conjecture 5 was established in [6], though Theorem 2 shows that this conjecture is not always true. We will present, in Section 4, further counterexamples to Conjecture 5.

In investigating the general 3-element sets $D = \{i, j, k\}$, Gupta [19] extended Theorem 3 to a similar formula for $D = \{i, j, k\}$ when j is odd. In addition, the author also established a lower bound of $\mu(D)$ for $D = \{i, j, k\}$ when j is even for most values of k , and showed that the bound is sharp for some cases. When $i = 1$, these results recover exactly the results of Theorems 2 and 3, thus leaving the same D -sets undetermined.

The parameters $\kappa(D)$ and $\mu(D)$ are closely related to coloring parameters of distance graphs (cf. [28]). We introduce some of these connections in the following section.

2 Distance Graphs and Preliminaries

Let D be a set of positive integers. The *distance graph generated by D* , denoted as $G(\mathbb{Z}, D)$, has the integers \mathbb{Z} as the vertex set. Two vertices are adjacent whenever the absolute value of their difference falls in D . The study of distance graph was introduced by Eggleton, Erdős, and Skilton [15] in 1985, and has been studied intensively ([5, 7, 15–17, 21–30, 39, 40, 42–44]).

The *fractional chromatic number* of a graph G , denoted by $\chi_f(G)$, is the minimum ratio m/n ($m, n \in \mathbb{Z}^+$) of an (m/n) -coloring, where an (m/n) -coloring is a function on $V(G)$ to n -element subsets of $[m] = \{1, 2, \dots, m\}$ such that if $uv \in E(G)$ then $f(u) \cap f(v) = \emptyset$. Denote the fractional chromatic number of $G(\mathbb{Z}, D)$ by $\chi_f(D)$. Chang et al. [7] proved that, for any set of positive integers D , $\chi_f(D) = 1/\mu(D)$. This result brought a new insight in the study of distance graphs. Combining this with (1) we have

$$\frac{1}{\mu(D)} = \chi_f(D) \leq \frac{1}{\kappa(D)}. \quad (2)$$

Viewing the problem of determining $\mu(D)$ through the lens of fractional chromatic number can often yield simple and elegant results. For example, using (2), Chang et al. [7] simplified Cantor and Gordan's proof that $\mu(\{a, b\}) = \lfloor \frac{a+b}{2} \rfloor / (a+b)$ when $\gcd(a, b) = 1$ by showing that $G(\mathbb{Z}, D)$ contains as a subgraph an odd cycle of length

$a + b$. As we know the fractional chromatic number of an $(2n + 1)$ -cycle is $(2n + 1)/n$, this bounds the fractional chromatic number from below. It is also clear from this point of view that $\mu(D) = \mu(aD)$, where $aD = \{ad : d \in D\}$, by considering the fact that $G(\mathbb{Z}, aD)$ is just a disjoint, isomorphic copies of $G(\mathbb{Z}, D)$.

The following result, due to Haralambis [20], is one of the few tools for bounding $\mu(D)$ from above.

Lemma 6. [20] *Let D be a set of positive integers, and let $\alpha \in (0, 1]$. If for every D -sequence S with $0 \in S$ there exists a positive integer n such that $S[n]/(n + 1) \leq \alpha$, then $\mu(D) \leq \alpha$.*

We shall frequently use the following equivalent definition of $\kappa(D)$. For positive integers m and x , denote $\|x\|_m = \min\{x \pmod m, m - x \pmod m\}$. Let D be a set of positive integers and t a positive integer, denote $\|tD\|_m = \min\{\|td\|_m : d \in D\}$. Then

$$\kappa(D) = \max \left\{ \frac{\|tD\|_m}{m} : \gcd(m, t) = 1 \right\}.$$

Let a be a positive integer. It can be seen from the definition that $\kappa(aD) = \kappa(D)$. This, together with the fact shown above that $\mu(D) = \mu(aD)$, allows us to assume $\gcd(D) = 1$ unless mentioned otherwise. In addition, it can be easily seen that if D is a singleton or contains only odd numbers, then $\mu(D) = \kappa(D) = 1/2$. Finally, by definition, if $D' \subseteq D$, then $\mu(D') \leq \mu(D)$ and $\kappa(D') \leq \kappa(D)$.

3 $D = \{1, j, k\}$ and j is odd

We begin with a result that partially confirms Conjecture 4.

Theorem 7. *Let $D = \{1, j, mj\}$ with j odd, $j \geq 3$. Then $\mu(D) = \kappa(D) = 1/2$ if m is odd; and $\mu(D) = \kappa(D) = \frac{m}{2(m+1)}$ if m is even.*

Proof. We only need to show the case when m is even. Note, $\mu(D) \leq \mu(\{j, mj\}) = \mu(\{1, m\}) = \kappa(\{1, m\}) = \frac{m}{2(m+1)}$. It is enough to show that there exists some t with $\|tD\|_{mj+j} \geq mj/2$. One can see that $t = (mj + j - 1)/2$ would fulfill this requirement. \square

In the remaining of the section we show that Conjecture 4 does not always hold.

Theorem 8. *Let $D = \{1, j, k\}$ with $j = 4n + 3$, $k \equiv 2n \pmod{2n + 1}$, k even, and $k < (2n + 1)(4n + 3)$. Then $\mu(D) = \kappa(D) = \frac{n}{2n+1}$.*

Proof. First we claim $\kappa(D) \geq n/(2n+1)$. By hypotheses, we have $\|k\|_{2n+1} = \|j\|_{2n+1} = 1$, so $\|nD\|_{2n+1} = n$, and $\kappa(D) \geq n/(2n + 1)$.

Next we prove $\mu(D) \leq n/(2n+1)$. By (2), it is sufficient to show that $\chi_f(D) \geq \frac{2n+1}{n}$ by exhibiting a cycle of length $2n + 1$ as a subgraph in $G(D)$. Let $k = m(2n + 1) + 2n$. Note that since k is even, m is also even. Let $m' = m/2$. The value of m' is bounded above by $2n$ because $k \leq 2n(4n + 3)$. Hence the following vertices form a cycle of length $2n + 1$ in $G(D)$: $\{0, j, 2j, \dots, m'j, m'j + 1, m'j + 2, \dots, m'j + 2n - m' = k\}$. \square

The next two theorems make use of the following lemma.

Lemma 9. *Let $D = \{1, j, k\}$ with j odd, k even, and k not a multiple of j . If S is a D -sequence with $\{0, 2, \dots, j - 1\} \subset S$, then $S[j + k - 1] \leq k/2$.*

Proof. Let $k = qj + k^*$ where $1 \leq k^* \leq j - 1$. Partition $\{j, j + 1, \dots, j + k - 1\}$ into the residue classes modulo j :

$$A_i = \{j + i, 2j + i, \dots, m_i j + i\}, \quad 0 \leq i \leq j - 1, \quad m_i j + i \leq j + k - 1.$$

Note $j + k - 1 = j(q + 1) + k^* - 1$, so

$$m_i = |A_i| = \begin{cases} q + 1 & 0 \leq i \leq k^* - 1; \\ q & k^* \leq i \leq j - 1. \end{cases}$$

The fact that $j \in D$ and $\{0, 2, \dots, j - 1\} \subset S$ implies that $\min(A_i) \notin S$ if i is even. Also, we have

$$\max(A_i) = \begin{cases} (q + 1)j + i = k + j + i - k^* & 0 \leq i \leq k^* - 1; \\ qj + i = k + i - k^* & k^* \leq i \leq j - 1. \end{cases}$$

Together with the assumptions $k \in D$, j is odd, k is even, and $\{0, 2, \dots, j - 1\} \subset S$, we obtain

Observation. If $0 \leq i \leq k^* - 1$ and $i \not\equiv k^* \pmod{2}$, or if $k^* \leq i \leq j - 1$ and $i \equiv k^* \pmod{2}$, then $\max(A_i) \notin S$.

Now we consider two cases depending on the parity of q .

Case 1: Assume q is even. This implies k^* is even as well. By Observation, $\max(A_i) \notin S$, if $0 \leq i \leq k^* - 1$ and i is odd, or if $k^* \leq i \leq j - 1$ and i is even. Note as $j \in D$, for every two consecutive elements in A_i at most one is in S . Thus, in conclusion, we have

$$|A_i \cap S| \leq \begin{cases} \frac{q}{2} & k^* \leq i \leq j - 1 \text{ and } i \text{ odd, or } 0 \leq i \leq k^* - 1; \\ \frac{q-2}{2} & k^* \leq i \leq j - 1 \text{ and } i \text{ even.} \end{cases}$$

And

$$S[j + k - 1] \leq \frac{j+1}{2} + \frac{q}{2} \left(k^* + \frac{j - k^* - 1}{2} \right) + \left(\frac{q-2}{2} \right) \left(\frac{j - k^* + 1}{2} \right) = \frac{k}{2}.$$

Case 2: Assume q is odd. This implies k^* is odd as well. By Observation, $\max(A_i) \notin S$, if $0 \leq i \leq k^* - 1$ and i is even, or if $k^* \leq i \leq j - 1$ and i is odd. Thus, in conclusion, we have

$$|A_i \cap S| \leq \begin{cases} \frac{q-1}{2} & 0 \leq i \leq k^* - 1 \text{ and } i \text{ even, or } k^* \leq i \leq j - 1; \\ \frac{q+1}{2} & 0 \leq i \leq k^* - 1 \text{ and } i \text{ odd.} \end{cases}$$

And

$$S[j + k - 1] \leq \frac{j+1}{2} + \left(\frac{q-1}{2} \right) \left(\frac{k^* + 1}{2} + j - k^* \right) + \left(\frac{q+1}{2} \right) \left(\frac{k^* - 1}{2} \right) = \frac{k}{2}.$$

□

Theorem 10. Let $D = \{1, j, k\}$ with $j = 4n + 3$, $k \equiv 2n + 2 \pmod{4n + 4}$, and $k \leq j(j - 1)/2$. Then $\mu(D) = \kappa(D) = \frac{2n+1}{4n+4}$.

Proof. Since $k \equiv 2n + 2 \pmod{4n + 4}$ and $j \equiv -1 \pmod{4n + 4}$, one can easily check that $|(2n + 1)D|_{4n+4} \geq 2n + 1$. Thus $\kappa(D) \geq (2n + 1)/(4n + 4)$.

Next we show $\mu(D) \leq (2n + 1)/(4n + 4)$. If $S[j] \leq (4n + 4)(2n + 1)/(4n + 4) = 2n + 1$ holds for every D -sequence S with $0 \in S$, then by Lemma 6, the result follows. Thus assume there exists a D -sequence S with $0 \in S$ and $S[j] \geq 2n + 2$. This implies that $\{0, 2, \dots, j - 1\} \subset S$.

Let $k = q(4n + 4) + 2n + 2$. By assumption $k \leq j(j - 1)/2$, it must be $1 \leq q \leq 2n$. By Lemma 6, it suffices to show that

$$S[j + k - 1] \leq (j + k)(2n + 1)/(4n + 4) = 2nq + q + 3n + 1 + \frac{1}{4n + 4}.$$

Since $k = q(4n + 4) + 2n + 2$ and $q \leq 2n$, by Lemma 9, we have $S[j + k - 1] \leq 2nq + 2q + n + 1 \leq 2nq + q + 3n + 1$, as required. □

Theorem 11. *Let $D = \{1, j, k\}$ with $j = 4n + 1$, k even, $k \equiv \pm 1 \pmod{2n + 1}$, and $k \leq 2n(4n + 1)$. Then $\mu(D) = \kappa(D) = \frac{n}{2n+1}$.*

Proof. Since $k \equiv \pm 1 \pmod{2n + 1}$ and $j \equiv -1 \pmod{2n + 1}$, one can easily check that $\|nD\|_{2n+1} \geq n$. Thus $\kappa(D) \geq n/(2n + 1)$. It remains to show $\mu(D) \leq n/(2n + 1)$. We proceed by considering two cases.

Case 1: $k \equiv 1 \pmod{2n+1}$. Write $k = (2q+1)(2n+1)+1$ where $0 \leq q \leq 2n-1$. Let S be a D -sequence with $0 \in S$. Note that if $S[j] \leq 2n$, by Lemma 6 the results follow. So we assume $S[j] \geq 2n+1$. Since $1 \in D$ and $0 \in S$, it must be that $\{0, 2, \dots, j-1\} \subset S$. By Lemma 6, it suffices to show

$$S[j+k-1] \leq (j+k)n/(2n+1) = 2nq + 3n.$$

Since $k = (2q+1)(2n+1)+1$ and $q \leq 2n-1$, by Lemma 9, we have $S[j+k-1] \leq 2nq + q + n + 1 \leq 2nq + 3n$, as required.

Case 2: $k \equiv 2n \pmod{2n+1}$. The proof is similar to Case 1. Let $k = (2q+1)(2n+1)-1$ where $1 \leq q \leq 2n-1$. If $S[j] \leq 2n$, by Lemma 6 we are done. So we assume $S[j] = 2n+1$, which implies $\{0, 2, \dots, j-1\} \subset S$. By Lemma 6, it suffices to show

$$S[j+k-1] \leq 2nq + 3n - 1.$$

Since $k = (2q+1)(2n+1)-1$ and $q \leq 2n-1$, by Lemma 9, we have $S[j+k-1] \leq 2nq + q + n \leq 2nq + 3n - 1$, as required. \square

Note Theorems 8, 10 and 11 show that the assumption $k \geq j(j-1)/2$ in Theorem 3 is sharp.

4 $D = \{1, j, k\}$ and j is even

Now we consider $D = \{1, j, k\}$ when j is even. Theorem 2 applies to most D sets in this family, in the sense that for a given even integer j there are only a finite number of integers k such that $D = \{1, j, k\}$ is not covered by the theorem. But the restrictions of Theorem 2 on k are not simple bounds, so those k for which $\mu(D)$ is still unknown are not all consecutive. Considering the case when k is a multiple of j , direct calculation shows that Theorem 2 determines the values of $\mu(D)$ when k is an odd multiple of j .

Next we determine the value of $\mu(D)$ when $k = mj$ and m is even. In this case, we obtain the upper bound $\mu(\{1, j, mj\}) \leq \mu(\{j, mj\}) = \frac{m}{2(m+1)}$. On the other hand, to get a lower bound for $\kappa(D)$ the “good time” t approach used in the proof of Theorem 7 no longer works. In this case it is easier to find the kappa value using the following lemma which holds for general 3-element sets D .

Lemma 12. *Let $D = \{i, j, k\}$, where $\gcd(D) = 1$ and $\gcd(i, j) = d$. If $\frac{i}{d} + \frac{j}{d} = 2x + 1$ and $d \geq 2x$, then $\kappa(D) = \frac{x}{2x+1}$.*

Proof. First note that $\frac{x}{2x+1} = \frac{(i+j-d)/2}{i+j} = \kappa(\{i, j\})$ as proved in [4]. So it remains to show that $\kappa(D) \geq \frac{x}{2x+1}$.

By hypothesis $2x + 1$ and i/d are relatively prime, so let a be the unique solution to the congruence equation

$$a \left(\frac{i}{d} \right) \equiv x \pmod{2x+1}.$$

Multiplying this equation by d gives:

$$ai \equiv \frac{i+j-d}{2} \pmod{i+j}.$$

This together with the fact that $j \equiv -i \pmod{i+j}$ implies:

$$aj \equiv \frac{i+j+d}{2} \pmod{i+j}.$$

Furthermore, for every positive integer n ,

$$|(a + n(2x+1))i|_{i+j} = |(a + n(2x+1))j|_{i+j} = \frac{i+j-d}{2}.$$

Thus, in order to show that $\kappa(D) \geq \frac{(i+j-d)/2}{i+j}$, it suffices to show there exists some n such that $|(a + n(2x+1))k|_{i+j} \geq \frac{i+j-d}{2}$.

Let $ak \equiv c \pmod{2x+1}$ for some $0 \leq c \leq 2x$. Clearly $(a + n(2x+1))k \equiv c \pmod{2x+1}$ for all n . The residue classes of $(a + n(2x+1))k$ modulo $i+j$ will range over the entire set $\{c + m(2x+1) \mid 0 \leq m \leq d-1\}$ as n ranges from 0 to $d-1$. To show this, assume p and q be distinct integers with $0 \leq p \leq q \leq d-1$ and

$$k(a + p(2x+1)) \equiv k(a + q(2x+1)) \pmod{i+j}.$$

Then, $k(q - p) = ld$ for some positive integer l . This is impossible as $(q - p) < d$ and $\gcd(k, d) = 1$.

Because $\frac{i+j+d}{2} - \frac{i+j-d}{2} = d$, by the hypothesis $d \geq 2x + 1$, there exists some $m \in \{0, 1, \dots, d - 1\}$ such that $\frac{i+j-d}{2} \leq c + m(2x + 1) \leq \frac{i+j+d}{2}$, as needed. Thus, $\kappa(D) \geq x/(2x + 1)$. \square

Theorem 13. *Let $D = \{1, j, mj\}$ with both j and m even and $m \leq j$. Then $\mu(D) = \kappa(D) = \frac{m}{2(m+1)}$.*

Proof. First, $\mu(D) \leq \mu(\{j, mj\}) = \mu(\{1, m\}) = \kappa(\{1, m\}) = \frac{m}{2(m+1)}$. When $m \leq j$, by the previous lemma $\mu(D) \geq \kappa(D) = \frac{m}{2(m+1)}$. \square

Note, when $k = mj$ and m is even, Theorem 2 applies for $m \geq j - 2$. Thus, Theorems 2 and 13 together settle the family $D = \{1, j, k\}$ where k is an even multiple of j . When $m \in \{j - 2, j\}$, both Theorems 2 and 13 apply. But when $m \leq j - 4$, Theorem 13 gives a different value than Theorem 2 would without the boundary condition on k . This implies the boundary condition in Theorem 2 is sharp when k is a multiple of j .

It is interesting to compare Theorem 7 with Theorem 13. When j is odd, the subset $\{j, mj\}$ always determines the density of $\{1, j, mj\}$, but when j is even it only works when $m \leq j$. This is because when j is odd $\mu(\{1, j\}) = \frac{1}{2}$, thus the subset $\{1, j\}$ never forces the value of $\mu(D)$ lower than $\mu(\{j, mj\})$. But when j is even and $m > j$, the most restrictive two element subset of D changes since $\mu(\{1, j\}) < \mu(\{j, mj\})$. Note, when $m > j$, the subset $\{1, j\}$ does not always determine the density. Instead Theorem 2 applies.

5 Computations and Future Study

Computing the exact value of $\mu(D)$ for particular D -sets is a difficult task. One can refashion the contrapositive of Haralambis' lemma (Lemma 6) into an algorithm to search for an n such that no D -sequence of length n has the required density, thus confirming an upper bound on $\mu(D)$. This together with a suitably dense sequence can prove the exact value of $\mu(D)$, though it clearly becomes computationally intractable quickly.

Another method, the one developed by Carraher et al.[6] (using a result of Lih et al. [25]), is to squeeze the value of $\mu(D)$ between the independence ratio of the circulant graph on \mathbb{Z}_n and the independence ratio of the subgraph of the distance graph induced by $[m] = \{1, 2, \dots, m\}$:

$$\frac{\alpha(G(\mathbb{Z}_n, D))}{n} \leq \mu(D) = \frac{1}{\chi_f(D)} \leq \frac{\alpha(G(\mathbb{Z}, D)[m])}{m}. \quad (3)$$

Recall that $\alpha(G)$ is the independence number of a graph G , the maximum cardinality of an independent set of vertices in G . The *circulant graph* $G(\mathbb{Z}_n, D)$ has the integers modulo n as the vertex set with two vertices are adjacent if their difference modulo n is in D . Finding an m and an n such that the bounds in (3) are equal determines the exact value of $\mu(D)$. But this method is also computationally intense, as it involves the NP-hard task of finding maximum independent sets.

In the face of such difficulty, a conjecture of Haralambis is particularly interesting.

Conjecture 14. *For all sets of positive integers D such that $|D| \leq 3$, $\kappa(D) = \mu(D)$.*

The conjecture is true for $|D| < 3$, and when $|D| = 4$, Haralambis [20] and Liu and Zhu [31] produced D -sets such that $\mu(D) > \kappa(D)$, which can be extended to D -sets of any cardinality greater than 4. But when $|D| = 3$ the conjecture remains open. Computer calculations using the first method listed above have confirmed the conjecture for all three element D -sets with elements no greater than 25.

Conjecture 14 provides a tempting and so far productive approach to estimating the value of $\mu(D)$, as the value of $\kappa(D)$ can be computed comparatively easily. The authors identified the families of D -sets involved in this paper using the values of $\kappa(D)$.

As a final note, this paper is in large part a validation of the perspicacity of Haralambis, whose paper has been a fertile inspiration to the authors. Haralambis was able to discover sharp bounds for his theorems without (we imagine) access to the large amount of computer generated data available to us today. Verification of his 40 year old conjecture would be a worthy result. It seems likely that new methods will be required in a successful approach to the problem.

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